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A Convexity Property of Discrete Random Walks

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We establish a convexity property for the hitting probabilities of discrete random walks in $\mathbb{Z}^d$ (discrete harmonic measures). For $d = 2$ this implies a recent result on the convexity of the density of certain harmonic measures.

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1. The result in the plane

Let $\mathbb{Z}$ be the set of the integers and $\mathbb{Z}^2$ the integer lattice in the plane. We identify $\mathbb{Z}$ with $\mathbb{Z} \times \{0\}$ in $\mathbb{Z}^2$. In this paper we first prove a discrete result.

Theorem 1.1. *Let $X \subset \mathbb{Z}$ be any subset of the integers with $0 \notin X$, and start a symmetric random walk on $\mathbb{Z}^2$ from the origin which terminates when it hits a point of X. Let P_k be the probability that the walk terminates at $k \in X$. Then, for $k-1, k, k+1 \in X$, we have $P_k \leqslant \frac{1}{2}(P_{k-1} + P_{k+1})$.*

For the higher-dimensional analogue of this see Section 4.

A (finite or infinite) sequence $\{a_k\}_{k\in S}$, where $S \subset X$ is such that if two numbers $k < l$ belong to S then every $k < s < l$ also belongs to S, is said to be *convex* if $a_k \leqslant \frac{1}{2}(a_{k-1} + a_{k+1})$ for all k for which $k \pm 1 \in S$. In a standard way this implies that

$$a_k \leqslant \frac{s}{r+s} a_{k-r} + \frac{r}{r+s} a_{k+s} \tag{1.1}$$

provided $k - r, k + s \in S$. With this terminology, Theorem 1.1 implies the convexity of the hitting probabilities P_k, $k \in S$, for all S that consist of consecutive numbers in X. For example, it immediately follows that if all the integers in the interval $[k - m, k + m]$ lie in X, then $P_k \leqslant 1/(2m + 1)$ (m is an integer), as can be seen from $P_{k-m} + \cdots + P_{k+m} \leqslant 1$ and $P_{k-i} + P_{k+i} \geqslant 2P_k$, $i = 1, \ldots, m$. Another immediate consequence is that all level sets $\{k \mid P_k \leqslant \alpha\}$, $0 < \alpha < 1$, intersect any interval of X in an interval, where in this context an *interval* means a set of consecutive integers. Hence if X consists of $m \geqslant 1$ intervals, then every level set $\{k \mid P_k \leqslant \alpha\}$ consists of at most m intervals.

Theorem 1.1 was motivated by the paper [2] on the convexity of the density of harmonic measures (see the discussion below). There is a vast literature on discrete random walks; they are of primary importance, not just in probability theory but also in combinatorics, discrete potential theory/harmonic analysis, electrical network theory and statistical physics. In some cases the discrete models help to explain the continuous ones, but in some other cases the continuous versions are easier to handle. This is the situation in the present case, when there are explicit analytic formulas for continuous harmonic measures, which are not available in the discrete setting. Therefore, we believe that Theorem 1.1 and its higher-dimensional analogue Theorem 4.1, simple as they look, are of interest.

The theorem is strong enough to imply a recent result on harmonic measures. Let G be a domain in the plane such that its boundary ∂G consists of a finite number of Jordan curves and arcs. If $J \subset \partial G$ is a Jordan subarc of the boundary and $z_0 \in G$ is a fixed point, then let $\omega(z_0, J; G)$ be the harmonic measure of J with respect to z_0: $\omega(z_0, J; G)$ is the value $g(z_0)$ of the function g that is harmonic in G, and on the boundary takes the value 1 on J and 0 on $\partial G \setminus J$ (see [1, 6, 10] for the concept of harmonic measures, and in particular for the existence of g as a solution of a generalized Dirichlet problem). Harmonic measures play a fundamental role in harmonic analysis. For example, they are the representing measures for harmonic functions: if u is harmonic in G and continuous on its closure, then the so-called Poisson representation

$$u(z) = \int_{\partial G} u \, \mathrm{d}\omega(z, \cdot, G)$$

holds.

An alternative definition is as follows (see [5, 6]). Start a Brownian motion B at z_0 and let $P_{z_0}(J)$ be the probability that B hits the boundary ∂G of G first at a point of J. Then $\omega(z_0, J; G) = P_{z_0}(J)$. See [5] or [9] for more on the connection of probability theory and harmonic analysis.

In this terminology the sequence $\{P_k\}_{k \in X}$ from Theorem 1.1 is the discrete harmonic measure in $\mathbb{Z}^2 \setminus X$ with respect to the point 0.

As an illustration of Theorem 1.1 we derive the following continuous result.

Corollary 1.2. *If $E \subset \mathbb{R}$ consists of finitely many intervals and $z \in \mathbb{R} \setminus E$, then the harmonic measure $\omega(z, \cdot; \overline{\mathbb{C}} \setminus E)$ is absolutely continuous on E and its density is convex on any subinterval of E.*

This easily implies its more general form when E is any compact subset of the real line, and convexity of the density is claimed on any interval that is contained in E. In particular, if E is compact, then the density of the equilibrium measure (see [1, 6] or [10] for the definition) of E is convex on every subinterval of E, because the equilibrium measure is simply $\omega(\infty, \cdot; \mathbb{C} \setminus E)$.

Corollary 1.2 was proved in the recent paper [2] using iterated balayage. Theorem 1.1, which can be considered as its discrete version, gives another proof.

In Sections 2 and 3 we prove Theorem 1.1 and Corollary 1.2 modulo a technical statement, the proof of which can be found in the Appendix. In Section 4 we discuss the higher-dimensional analogue of Theorem 1.1.

2. Proof of Theorem 1.1

The proof is based on the following lemma.

Lemma 2.1. *Let p_k be the probability that a symmetric random walk on the integer lattice $\mathbb{Z}^2$ starting from the point $(0, 1)$ first hits the x-axis at the point $x = k$. Then $\{p_k\}_{k=0}^{\infty}$ is a convex sequence.*

We prove Lemma 2.1 later in this section. First we show how Theorem 1.1 follows from it.

Proof of Theorem 1.1. Start a random walk on $\mathbb{Z}^2$ from the origin, and let q_k be the probability that, after leaving the origin, the walk hits the x-axis first at the point $k \in \mathbb{Z}$. Then $q_1 = q_{-1} = \frac{1}{4} + \frac{1}{2}p_1$, but for all other k (including $k = 0$) we have $q_k = \frac{1}{2}p_k$, since to hit any $k \neq \pm 1$ before first hitting any other point on the real line, the walk has to move either to $(0, 1)$ or to $(0, -1)$, and the probability of first hitting $k \in \mathbb{Z}$ from there is p_k. Hence, together with the sequence $p_0, p_1, p_2, \dots$, the (identical) sequences $q_1, q_2, q_3, \dots$ and $q_{-1}, q_{-2}, q_{-3}, \dots$ are also convex.

Any walk (from the origin) terminating at a point of X can visit the points of $\mathbb{Z} \setminus X$ a few times. The probability that a walk is terminated at $k \in X$ having previously visited precisely the points $j_1, \dots, j_s \in \mathbb{Z} \setminus X$ in that order (where $j_i = j_{i+1}$ is possible if in the meantime the walk leaves the x-axis) is clearly $q_{j_1} q_{j_2 - j_1} \cdots q_{k - j_s}$; hence

$$P_k = \sum_{s \in \mathbb{N},\ j_1, j_2, \dots, j_s \in \mathbb{Z} \setminus X} q_{j_1} q_{j_2 - j_1} \cdots q_{k - j_s}.$$

The same formula is true for P_{k-1} and P_{k+1} (replacing k with $k \pm 1$), and since

$$k - 1 - j_s, k - j_s, k + 1 - j_s$$

are either all positive or all negative (note that $k - 1, k, k + 1 \in X$ while $j_s \notin X$), the convexity of the sequence $\{q_k\}_{k=1}^{\infty} = \{q_k\}_{k=-\infty}^{-1}$ gives that

$$q_{j_1} q_{j_2 - j_1} \cdots q_{k - j_s} \leqslant \frac{1}{2}(q_{j_1} q_{j_2 - j_1} \cdots q_{k - 1 - j_s} + q_{j_1} q_{j_2 - j_1} \cdots q_{k + 1 - j_s}).$$

Summing this for all s and $j_1, \dots, j_s \in \mathbb{Z} \setminus X$, we obtain the result. □

Let $\widetilde{p}_k$ be the probability that a discrete random walk starting from the point $(0,2)$ first hits the x-axis at the point $k \in \mathbb{Z}$. Clearly $p_{-k} = p_k$, and since the walk from $(0,1)$ can move to the points $(0,0)$, $(0,2)$, $(-1,1)$ and $(1,1)$, we also have

$$p_k = \frac{1}{4}(p_{k-1} + p_{k+1} + \widetilde{p}_k) \tag{2.1}$$

if $k \neq 0$. For $\tilde{p}_k$ the key estimate is contained in the next statement.

Lemma 2.2. *For all integers k,*

$$\widetilde{p}_k \leqslant p_{k-1} + p_{k+1}. \tag{2.2}$$

This immediately implies Lemma 2.1.

Proof of Lemma 2.1. For $k \geqslant 1$ the required inequality $p_k \leqslant \frac{1}{2}(p_{k-1} + p_{k+1})$ follows from (2.1) and (2.2). □

To complete the proof of Theorem 1.1, we still need to prove Lemma 2.2.

Proof of Lemma 2.2. Let $p_{k,l}$ be the probability that a random walk on $\mathbb{Z}^2$ starting from the point $(0,1)$ first hits the x-axis at the point $x = k$, and this hit occurs at the lth step, and let $\widetilde{p}_{k,l}$ be the same probability, but for a walk that starts from the point $(0,2)$. Since

$$p_k = \sum_{l=1}^{\infty} p_{k,l}$$

and

$$\widetilde{p}_k = \sum_{l=1}^{\infty} \widetilde{p}_{k,l},$$

it is enough to prove that

$$\widetilde{p}_{k,l} \leqslant p_{k-1,l} + p_{k+1,l} \tag{2.3}$$

holds for all k and l.

We let $\mathcal{W}_{k,l}$ denote the set of $(l-1)$-step (non-random) walks on $\mathbb{Z}^2$ from the point $(0,1)$ to the point $(k,1)$ that never hit the x-axis, and we let $\widetilde{\mathcal{W}}_{k,l}$ be the set of $(l-1)$-step walks on $\mathbb{Z}^2$ from the point $(0,2)$ to the point $(k,1)$ that never hit the x-axis. Then $p_{k,l} = |\mathcal{W}_{k,l}|(1/4)^l$ and $\widetilde{p}_{k,l} = |\widetilde{\mathcal{W}}_{k,l}|(1/4)^l$, so in order to prove (2.3), it is enough to show that

$$|\widetilde{\mathcal{W}}_{k,l}| \leqslant |\mathcal{W}_{k-1,l}| + |\mathcal{W}_{k+1,l}|. \tag{2.4}$$

The existence of an injective function $\widetilde{\mathcal{W}}_{k,l} \to \mathcal{W}_{k-1,l} \cup \mathcal{W}_{k+1,l}$ obviously implies (2.4), so we now give such a function ϕ.

Before proceeding, we suggest the reader think of the walks in $\mathcal{W}_{k-1,l}$ as $(1,1) \to (k,1)$ walks (after a translation to the right) and of the walks in $\mathcal{W}_{k+1,l}$ as $(-1,1) \to (k,1)$ walks

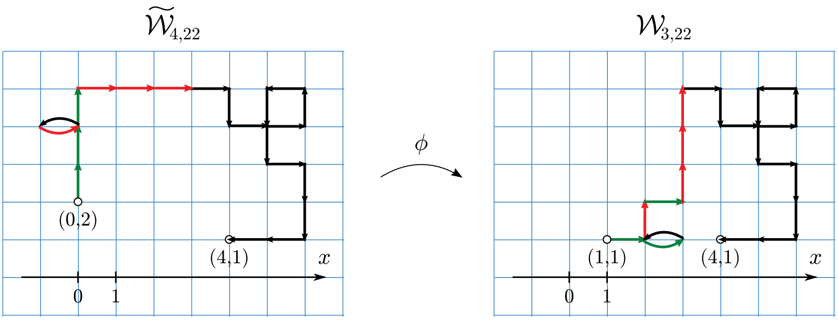

Figure 1. Illustration of ϕ, Case 1.

(after a translation to the left). From now on, we also apply this trivial redefinition of the sets $\mathcal{W}_{k-1,l}$ and $\mathcal{W}_{k+1,l}$. Pick an arbitrary walk $W \in \widetilde{\mathcal{W}}_{k,l}$. If W starts with a right-step, it seems natural to replace it with an up-step to get a walk in $\mathcal{W}_{k-1,l}$. Similarly, if W starts with a down-step, it seems natural to replace it with a right-step to get a walk in $\mathcal{W}_{k+1,l}$. We just generalize these ideas with the help of some kind of reflection. Now we present the definition of the image of W.

Let t be the smallest natural number for which it is true that in the first t steps of W there are more right-steps than up-steps (Case 1) or there are more down-steps than right-steps (Case 2). (The tth step is a right-step in Case 1, and it is a down-step in Case 2.) Such a t exists, because otherwise the number of right-steps would never exceed the number of up-steps and the number of down-steps would never exceed the number of right-steps, contradicting the fact that W contains more down-steps than up-steps.

In Case 1, we define $\phi(W)$ to be the walk W_1 that starts from the point $(1, 1)$, whose steps are obtained from the steps of W by replacing the right-steps with up-steps and the up-steps with right-steps among the first t steps, leaving the rest unchanged: see Figure 1. W_1 clearly has $l - 1$ steps, and it ends at the required point $(k, 1)$ because it contains one less right-step and one more up-step than W. The same reasoning shows that W_1 coincides with W after the tth step, so W_1 never hits the x-axis after the tth step. By the definition of t, for all $s \leqslant t$, in the first s steps of W there are at most as many down-steps as right-steps, and thus in the first s steps of W_1 there are at most as many down-steps as up-steps, that is, W_1 does not hit the x-axis in the first t steps either. This means that $W_1 \in \mathcal{W}_{k-1,l}$, that is, the above definition of $\phi(W)$ makes sense.

In Case 2, we define $\phi(W)$ to be the walk W_2 that starts from the point $(-1, 1)$, whose steps are obtained from the steps of W by replacing the down-steps with right-steps and the right-steps with down-steps among the first t steps, leaving the rest unchanged: see Figure 2. An analogous argument to that above shows that $W_2 \in \mathcal{W}_{k+1,l}$, that is, this definition also makes sense.

It is easy to see that ϕ is injective. For example, for a walk of $\mathcal{W}_{k-1,l}$, the unique inverse image, if it exists, can be found by interchanging the up-steps and right-steps until the number of up-steps exceeds the number of right-steps. The details are left to the reader. □

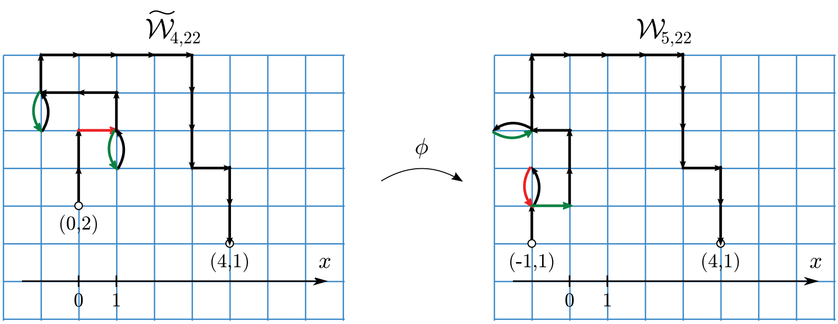

Figure 2. Illustration of ϕ, Case 2.

Remark 1. Since ϕ leaves the left-steps fixed, the following strengthening of (2.4) is also true. For all $L \subseteq \{1, \dots, l-1\}$,

$$|\widetilde{\mathcal{W}}^L_{k,l}| \leqslant |\mathcal{W}^L_{k-1,l}| + |\mathcal{W}^L_{k+1,l}|, \tag{2.5}$$

where

$$\mathcal{W}^L = \{W \in \mathcal{W} \,|\, \text{the } s\text{th step of } W \text{ is a left-step, if and only if } s \in L\}.$$

The cardinalities in (2.5) can be easily calculated explicitly, using the fact that Dyck paths are counted by the Catalan numbers. This yields a second proof of (2.5) and Lemma 2.1; see also the paper [7]. We have opted for the combinatorial proof given above since it does not involve any calculations.

Remark 2. Besides (2.1) we also have

$$p_0 = \frac{1}{4}(1 + p_{-1} + p_1 + \widetilde{p}_0).$$

Furthermore

$$\widetilde{p}_k = \sum_j p_j p_{k-j},$$

because a path from $(0, 2)$ to $(k, 0)$ must pass through a point $(j, 1)$. Hence, if

$$g(x) = \sum_k p_k e^{ikx},$$

then we have

$$g(x) = \frac{1}{4} + \frac{1}{4}(e^{ix} + e^{-ix})g(x) + \frac{1}{4}g^2(x).$$

The solution of this equation is

$$2 - \cos x \pm \sqrt{(1 - \cos x)(3 - \cos x)},$$

and we need the minus sign of the square root for $|g(x)| \leqslant 1$ for all x. Hence

$$g(x) = 2 - \cos x - \sqrt{(1 - \cos x)(3 - \cos x)}.$$

Now $p_k - (p_{k-1} + p_{k+1})/2$ is the Fourier coefficient in front of $\cos kx$ in

$$g(x)(1 - \cos x) = \frac{5}{2} - 3\cos x + \frac{1}{2}\cos 2x - (1 - \cos x)^{3/2}(3 - \cos x)^{1/2}.$$

For $k = 1, 2$ these are

$$-3 - \frac{2}{\pi}\int_0^\pi (1 - \cos x)^{3/2}(3 - \cos x)^{1/2}\cos x \,\mathrm{d}x = -3 + 2.84883\cdots < 0$$

and

$$\frac{1}{2} - \frac{2}{\pi}\int_0^\pi (1 - \cos x)^{3/2}(3 - \cos x)^{1/2}\cos 2x \,\mathrm{d}x = 0.5 - 0.546479 < 0$$

respectively. Thus, in view of the fact that g is even, the claim in Lemma 2.1 is equivalent to the positivity of

$$\gamma(k) = \int_0^\pi (1 - \cos x)^{3/2}(3 - \cos x)^{1/2}\cos kx \,\mathrm{d}x \tag{2.6}$$

for $k \geqslant 3$. This is possible to prove by an asymptotic analysis, although there is no easy way to see that the Fourier coefficients of a given function are positive. On the contrary, the easiest way to prove the positivity of the $\gamma(k)$ is via an independent proof of Lemma 2.1, as we have just done.

3. Proof of Corollary 1.2

We may assume that $z = 0$. Let $E \subset \mathbb{R}$ be the union of finitely many closed intervals, $0 \notin E$, and let $I \subset E$ be a subinterval of E. Now make the lattice of the walk denser: we make the walk on the lattice $(\varepsilon\mathbb{Z}) \times (\varepsilon\mathbb{Z})$, and let $P_\varepsilon(I)$ be the probability that this random walk hits E first in a point of I. Under proper normalization (it is convenient to use $\varepsilon = 1/\sqrt{n}$) this ε-walk tends to the standard Brownian motion B in the plane starting at the origin as $\varepsilon \to 0$, and $P_\varepsilon(I)$ tends to the probability that B hits E for the first time in a point of I. Since this latter probability is $\omega(0, I; \mathbb{C} \setminus E)$, we get

$$\lim_{\varepsilon \to 0} P_\varepsilon(I) = \omega(0, I; \mathbb{C} \setminus E), \quad \varepsilon = 1/\sqrt{n},\ n = 1, 2, \ldots. \tag{3.1}$$

Such limit relations go back about a century; see the paper [4] and the references therein. However, they are not explicitly about hitting probabilities as in our case, so we sketch a rigorous proof of (3.1) in the Appendix.

Suppose now that I has rational endpoints and d is a rational number such that $I - d$ and $I + d$ both belong to the same subinterval $J = [a, b]$ of E. Theorem 1.1 gives the convexity of the sequence $P_\varepsilon(\{k\varepsilon\})$, $k\varepsilon \in J$, and hence, if ε is such that both $|I|$ and d are integral multiples of ε, we get (*cf.* (1.1))

$$P_\varepsilon(I) \leqslant \frac{1}{2}(P_\varepsilon(I - d) + P_\varepsilon(I + d)).$$

On letting ε tend to 0 (if δ and the endpoints of I are of the form p/q with the same q, then we may set $\varepsilon = 1/\sqrt{N^2 q^2}$, $N = 1, 2, \ldots$, in (3.1) and here), we can conclude

$$\omega(0, I; \mathbb{C} \setminus E) \leqslant \frac{1}{2}\big(\omega(0, I - d; \mathbb{C} \setminus E) + \omega(0, I + d; \mathbb{C} \setminus E)\big). \tag{3.2}$$

Adding these for $d = |I|, 2|I|, \ldots, k|I|$, where k is the largest natural number for which $I - k|I|, I + k|I| \subseteq J$, we obtain

$$k\omega(0, I; \mathbb{C} \setminus E) \leqslant \omega(0, J; \mathbb{C} \setminus E).$$

Now if $J = [a, b]$ and $I \subset [a + \delta, b - \delta]$, then (for $|I| < \delta/4$) $k \geqslant \delta/2|I|$, and it follows that

$$\omega(0, I; \mathbb{C} \setminus E) \leqslant \frac{2}{\delta}|I|$$

(because $\omega(0, J; \mathbb{C} \setminus E) \leqslant 1$). This is true for intervals I with rational endpoints, and because $\omega(0, I; \mathbb{C} \setminus E)$ is monotone in I, the same inequality follows for all I. This shows the absolute continuity of $\omega(0, \cdot; \mathbb{C} \setminus E)$ on $[a + \delta, b - \delta]$ (with respect to the Lebesgue measure on $\mathbb{R}$), and hence on all of J. Let $v_E(t)$ denote the density of $\omega(0, \cdot; \mathbb{C} \setminus E)$ with respect to the Lebesgue measure, and let

$$v_E^*(x) = \limsup_{|I| \to 0,\ x \in I} \frac{\omega(0, I; \mathbb{C} \setminus E)}{|I|}.$$

This density v_E is determined only almost everywhere, but $v_E(x) = v_E^*(x)$ at every Lebesgue point of v_E, and hence almost everywhere. Hence, v_E^* can also be considered as the density of $\omega(0, \cdot; \mathbb{C} \setminus E)$ with respect to the Lebesgue measure, and we shall prove the convexity for v_E^*.

The convexity of the sequence $P_\varepsilon(\{k\varepsilon\})$, $k\varepsilon \in J$, implies more than just (3.2), namely with the same argument with which (3.2) was deduced, it also gives the stronger inequality (*cf.* (1.1))

$$\omega(0, I; \mathbb{C} \setminus E) \leqslant \frac{s}{r+s}\omega(0, I - r; \mathbb{C} \setminus E) + \frac{r}{r+s}\omega(0, I + s; \mathbb{C} \setminus E) \tag{3.3}$$

with positive rational $r, s, |I|$, so long as $I, I - r, I + s$ belong to J. The absolute continuity of $\omega(0, \cdot; \mathbb{C} \setminus E)$ then gives the same for all I (which may not have rational length). Now divide through in (3.3) by $|I|$, and, while keeping I above a given point $x \in J$ in the sense that $x \in I$, let $|I|$ tend to 0 through an appropriate sequence, so that $\omega(0, I; \mathbb{C} \setminus E)/|I|$ tends to $v_E^*(x)$. If at the same time $r/(r + s)$ tends to some $0 < \alpha < 1$ and r tends to some αy, then automatically s tends to $(1 - \alpha)y$; it follows that

$$v_E^*(x) \leqslant (1 - \alpha)v_E^*(x - \alpha y) + \alpha v_E^*(x + (1 - \alpha)y).$$

Hence v_E^* is convex on J, and since $v_E^* = v_E$ almost everywhere, the claim has been proved. □

4. Random walks in $\mathbb{Z}^d$

In this section we discuss the analogue of Theorem 1.1 in $\mathbb{Z}^d$. A point in $\mathbb{Z}^d$ has $2d$ neighbours, so in a symmetric random walk the probability of moving from a point to any one of its neighbours is $1/2d$. We shall identify $\mathbb{Z}^{d-1}$ with the sublattice $\mathbb{Z}^{d-1} \times \{0\}$, that is, with the set of points in $\mathbb{Z}^d$ for which the dth coordinate is 0. For $Q \in \mathbb{Z}^{d-1}$ let $\Sigma(Q)$ be the set of its $2(d - 1)$ neighbours in $\mathbb{Z}^{d-1}$. The analogue of Theorem 1.1 in $\mathbb{Z}^d$ is as follows.

Theorem 4.1. *Let $d \geqslant 2$, let $X \subset \mathbb{Z}^{d-1}$ be a subset of $\mathbb{Z}^{d-1}$ with $0 \notin X$, and start a symmetric random walk on $\mathbb{Z}^d$ from the origin which terminates when it hits a point of X. Let P_Q be the probability that the walk terminates at $Q \in X$. If a point Q and all its $2(d-1)$ neighbours in $\mathbb{Z}^{d-1}$ lie in X, then*

$$P_Q \leqslant \frac{1}{2(d-1)} \sum_{R \in \Sigma(Q)} P_R.$$

As in Section 1, this is a consequence of the following lemma (just repeat the argument after Lemma 2.1).

Lemma 4.2. *Let p_Q be the probability that a symmetric random walk on $\mathbb{Z}^d$ starting from the point $(0,\ldots,0,1)$ first hits $\mathbb{Z}^{d-1}$ at the point $Q \in \mathbb{Z}^{d-1}$. Then, for $Q \neq 0$, we have*

$$p_Q \leqslant \frac{1}{2(d-1)} \sum_{R \in \Sigma(Q)} p_R.$$

We note that although symmetric random walks in $\mathbb{Z}^d$ are not recurrent for $d \geqslant 3$, the probability that a walk starting from the point $(0,\ldots,0,1)$ hits $\mathbb{Z}^{d-1}$ is still 1.

Proof. Lemma 4.2. Let $\widetilde{p}_Q$ be the probability that a discrete random walk starting from the point $(0,\ldots,0,2)$ first hits $\mathbb{Z}^{d-1}$ at the point $Q \in \mathbb{Z}^{d-1}$. With this the analogue of (2.1) is

$$p_Q = \frac{1}{2d}\left(\sum_{R \in \Sigma(Q)} p_R + \widetilde{p}_Q\right)$$

for all $Q \neq 0$. Thus, the statement is derived from the following analogue of Lemma 2.2. For all $Q \in \mathbb{Z}^{d-1}$, we have

$$(d-1)\widetilde{p}_Q \leqslant \sum_{R \in \Sigma(Q)} p_R. \tag{4.1}$$

The set $\Sigma(Q)$ consists of $(d-1)$ disjoint pairs $\{Q_{i\pm}\}$, $1 \leqslant i \leqslant d-1$, where the point $Q_{i\pm}$ has the same coordinates as Q, except that its ith coordinate is obtained from the ith coordinate of Q by adding ± 1. Therefore, (4.1) will follow from the relation

$$\widetilde{p}_Q \leqslant p_{Q_{i+}} + p_{Q_{i-}} \tag{4.2}$$

that we prove for all $1 \leqslant i \leqslant d-1$. By symmetry, it is enough to consider $i = 1$.

As in the proof of Lemma 2.2, let $p_{Q,s}$ be the probability that a random walk on $\mathbb{Z}^d$ starting from the point $(0,\ldots,0,1)$ first hits $\mathbb{Z}^{d-1}$ at the point Q, and this hit occurs at the sth step. Let $\widetilde{p}_{Q,s}$ be the same probability for the walk that starts from the point $(0,\ldots,0,2)$. Since

$$p_Q = \sum_{s=1}^{\infty} p_{Q,s}$$

and

$$\widetilde{p}_Q = \sum_{s=1}^{\infty} \widetilde{p}_{Q,s},$$

it is sufficient to prove that

$$\widetilde{p}_{Q,s} \leqslant p_{Q_{1+},s} + p_{Q_{1-},s} \tag{4.3}$$

holds for all Q and s.

Let $Q = (Z_1, \dots, Z_{d-1})$, so that $Q_{1\pm} = (Z_1 \pm 1, Z_2, \dots, Z_{d-1})$, and we shall also use the notation $(Q,1)$ for the point $(Z_1, \dots, Z_{d-1}, 1)$ from $\mathbb{Z}^d$.

We let $\mathcal{V}_{Q,s}$ denote the set of $(s-1)$-step (non-random) walks V on $\mathbb{Z}^d$ from the point $(0, \dots, 0, 1)$ to the point $(Q,1)$ that never hit $\mathbb{Z}^{d-1}$, and similarly let $\widetilde{\mathcal{V}}_{Q,s}$ be the set of such $(s-1)$-step walks on $\mathbb{Z}^d$ from the point $(0, \dots, 0, 2)$ to the point $(Q,1)$. Then

$$p_{Q,s} = |\mathcal{V}_{Q,s}| \left(\frac{1}{2d}\right)^s, \quad \widetilde{p}_{Q,l} = |\widetilde{\mathcal{V}}_{Q,s}| \left(\frac{1}{2d}\right)^s,$$

and thus, in order to prove (4.3), it is enough to show that

$$|\widetilde{\mathcal{V}}_{Q,s}| \leqslant |\mathcal{V}_{Q_{1+},s}| + |\mathcal{V}_{Q_{1-},s}|, \tag{4.4}$$

which is the analogue of (2.4).

In the proof of Lemma 2.2 we were considering right/left and up/down steps. Instead of these we now have steps f_i/b_i (forwards/backwards along the x_i-axis), which increase/decrease the ith coordinate of a point by 1. Thus, a walk V in $\mathcal{V}_{Q,s}$ can be identified with a sequence $\tau_{i_1}, \dots, \tau_{i_{s-1}}$, where $i_j \in \{1, \dots, d\}$ and each τ is either f or b. Let

$$S = S(V) := \{j \mid i_j \in \{2, 3, \dots, d-1\}\}$$

be the places where V makes a move along one of the axis $x_2, \dots, x_{d-1}$. If

$$\sigma = \sigma(V) := \{\tau_{i_j} \mid j \in S(V)\}$$

(where we keep the original order of the τ_{i_j} from V), then clearly this σ is a walk from 0 to the point $(Z_2, \dots, Z_{d-1})$ in the integer lattice of $(x_2, \dots, x_{d-1})$, $x_i \in \mathbb{R}$, which we identify with the submanifold $(0, x_2, \dots, x_{d-1}, 0)$, $x_i \in \mathbb{R}$, of $\mathbb{R}^d$. For each such $S \subset \{1, \dots, s-1\}$ and $\sigma = \{\tau_1, \dots, \tau_{|S|}\}$, let $\mathcal{V}(Q, s; S, \sigma)$ be the set of walks V from $\mathcal{V}_{Q,s}$ for which $S(V) = S$ and $\sigma(V) = \sigma$, and $\widetilde{\mathcal{V}}(Q, s; S, \sigma)$ will be used analogously for $\widetilde{\mathcal{V}}_{Q,s}$. Then $\mathcal{V}_{Q,s}$ ($\widetilde{\mathcal{V}}_{Q,s}$) is a disjoint union of the sets $\mathcal{V}(Q, s; S, \sigma)$ ($\widetilde{\mathcal{V}}(Q, s; S, \sigma)$) for all possible S and σ that produce a walk from 0 to $(Z_2, \dots, Z_{d-1})$ in $(x_2, \dots, x_{d-1})$, $x_i \in \mathbb{Z}$. Hence (4.4) will follow if we prove

$$|\widetilde{\mathcal{V}}(Q, s; S, \sigma)| \leqslant |\mathcal{V}(Q_{1+}, s; S, \sigma)| + |\mathcal{V}(Q_{1-}, s; S, \sigma)|. \tag{4.5}$$

However, in all $V \in \mathcal{V}(Q, s; S, \sigma)$ ($V \in \widetilde{\mathcal{V}}(Q, s; S, \sigma)$) the movements in the $x_2, \dots, x_{d-1}$ directions are fixed and their number is $|S|$, so it is clear that, with the notation from the proof of Lemma 2.2,

$$|\widetilde{\mathcal{V}}(Q, s; S, \sigma)| = |\widetilde{\mathcal{W}}_{Z_1, s-|S|}|, \quad |\mathcal{V}(Q_{1\pm}, s; S, \sigma)| = |\mathcal{W}_{Z_1 \pm 1, s-|S|}|.$$

Hence (4.5) is a consequence of (2.4) with $k = Z_1$ and $l = s - |S|$. $\square$

Appendix: Proof of (3.1)

For the concepts used below on the Wiener measure and random walks, see any standard book such as those of Billingsley [3] or Kallenberg [8].

Let W be the Wiener measure on the space $C(\mathbb{R}_+)$ equipped with the topology of uniform convergence on compact sets, and let B_1, B_2 be standard independent Brownian motions on $\mathbb{R}_+$, that is, $B_j : \Omega \to C(\mathbb{R}_+)$ is a random function on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution W:

$$\mathbb{P}(B_j \in \mathcal{E}) = W(\mathcal{E})$$

for all Borel subsets $\mathcal{E}$ of $C(\mathbb{R}_+)$. Since we assumed that B_1 and B_2 are independent, (B_1, B_2) is a Brownian motion on the plane with distribution $W \times W$.

Let $\xi_1, \xi_2, \dots, \zeta_1, \zeta_2, \dots$ be independent variables with

$$\mathbb{P}(\xi_j = \pm 1/\sqrt{2}) = 1/2, \quad \mathbb{P}(\zeta_j = \pm 1/\sqrt{2}) = 1/2,$$

each of mean zero and variance 1. We set

$$x_{n,k} = \frac{1}{\sqrt{n}}(\xi_1 + \cdots + \xi_k), \quad x_n = (x_{n,k})_{k=1}^{\infty}.$$

Then x_n can be regarded as a symmetric random walk on $\mathbb{Z}/\sqrt{2n}$ whose position at time k/n is $x_{n,k}$. Let y_n be similarly generated from the ζ_j, so that x_n, y_n are independent discrete symmetric random walks on $\mathbb{Z}/\sqrt{2n}$. Let

$$\begin{aligned} X_n(t) &= x_{n,[nt]} + (nt - [nt])(x_{n,[nt]+1} - x_{n,[nt]}) \\ &= \frac{1}{\sqrt{n}}(\xi_1 + \cdots + \xi_{[nt]}) + (nt - [nt])\frac{\xi_{n,[nt]+1}}{\sqrt{n}}, \quad t \in \mathbb{R}_+ \end{aligned}$$

be the path of x_n, and let the function $Y_n \in C(\mathbb{R}_+)$ be defined similarly for y_n. Then (x_n, y_n) is a discrete symmetric random walk on the lattice $e^{i\pi/4}\mathbb{Z}^2/\sqrt{n}$, which is the lattice $\mathbb{Z}^2/\sqrt{n}$ rotated by 45 degrees. The function $(X_n, Y_n) \in C(\mathbb{R}_+) \times C(\mathbb{R}_+)$ is the path of this discrete walk. We may assume that the underlying probability space for x_n, y_n, X_n, Y_n is again $(\Omega, \mathcal{A}, \mathbb{P})$.

Since we have a discrete walk on the rotated lattice, we shall also need to assume that the set $0 \notin E$ consists of finitely many closed segments on the $x = y$ line, and $I \subset E$ is a closed subsegment of E. Let, as before (3.1), $P_{1/\sqrt{n}}(I)$ be the probability that (x_n, y_n) hits the set E first in a point of I, and let $P^*(I)$ be the same probability for the Brownian motion (B_1, B_2). Since the latter probability is $\omega(0, I, \mathbb{C} \setminus E)$, the limit (3.1) takes the form

$$\lim_{n\to\infty} P_{1/\sqrt{n}}(I) = P^*(I). \tag{A.1}$$

Let us denote weak convergence by $\Rightarrow$. According to Donsker's theorem on $\mathbb{R}_+$ ([8, Corollary 14.6]) we have $X_n \Rightarrow B_1$, $Y_n \Rightarrow B_2$ as $n \to \infty$, and since X_n, Y_n are independent, we also have $(X_n, Y_n) \Rightarrow (B_1, B_2)$ (see [3, Sec. 4, pp. 26–27]). Let $\mathcal{E}$ be the set of functions $f \in C(\mathbb{R}_+) \times C(\mathbb{R}_+)$ for which the first intersection of its trajectory with E occurs at a point of I, that is, $f \in C(\mathbb{R}_+) \times C(\mathbb{R}_+)$ belongs to $\mathcal{E}$ precisely if $f(t) \in E$ for some t, and $f(t_0) \in I$ is true for the smallest real number t_0 for which $f(t_0) \in E$. Since for sufficiently large n the path of the walk (x_n, y_n) on the rotated lattice can intersect a fixed segment on the $x = y$ line only if the walk itself hits that same segment, we obtain $P_{1/\sqrt{n}}(I) = \mathbb{P}((X_n, Y_n) \in \mathcal{E})$. At the same time

$$P^*(I) = W \times W(\mathcal{E}) = \mathbb{P}((B_1, B_2) \in \mathcal{E}).$$

Therefore, (A.1) follows if

$$\mathbb{P}((X_n, Y_n) \in \mathcal{E}) \to \mathbb{P}((B_1, B_2) \in \mathcal{E}), \quad n \to \infty$$

holds, which, in view of $(X_n, Y_n) \Rightarrow (B_1, B_2)$, is certainly true if the boundary $\partial\mathcal{E}$ of $\mathcal{E}$ has zero $(W \times W)$-measure ([3, Theorem 2.1]).

Let I be the segment $[A, B]$, let H_1 be the set of all $f \in C(\mathbb{R}_+) \times C(\mathbb{R}_+)$ which pass through A or B (*i.e.*, there is a $t \in \mathbb{R}_+$ with $f(t) = A$ or $f(t) = B$), and let H_2 be the set of all $f \in C(\mathbb{R}_+) \times C(\mathbb{R}_+)$ which touch I, that is, there are rational $0 < p < q$ and a point $t_0 \in (p, q)$ such that $f(t_0) \in I$, but on the interval (p, q) the point $f(t)$ is always on or above, or always on or below I: if $f(t) = (f_1(t), f_2(t))$, then either always $f_1(t) \leqslant f_2(t)$ or always $f_1(t) \geqslant f_2(t)$ on (p, q). According to which of these cases occur, we write $f \in H^+_{p,q}$ or $f \in H^-_{p,q}$, so

$$H_2 = \cup_{p<q\in\mathbb{Q}}(H^+_{p,q} \cup H^-_{p,q})$$

(here $\mathbb{Q}$ is the set of rational numbers). It is clear that $\partial\mathcal{E} \subset H_1 \cup H_2$, and that $W \times W(H_1) = 0$ (the probability that a two-dimensional Brownian motion passes through a given point different from the origin is 0). Thus, it is left to prove that both $H^+_{p,q}$ and $H^-_{p,q}$ have zero $(W \times W)$-measure for all $p < q$. But, for example,

$$W \times W(H^-_{p,q}) = \mathbb{P}((B_1, B_2) \in H^-_{p,q}),$$

and $(B_1, B_2) \in H^-_{p,q}$ means that the maximum of $B_1(t) - B_2(t)$ over the interval (p, q) is 0. Since $(B_1 - B_2)/\sqrt{2}$ is again a standard Brownian motion and the maximum of a Brownian motion on an interval has continuous distribution ([3, (10.17)]), the event

$$\max_{t\in(p,q)}(B_1(t) - B_2(t)) = 0$$

does indeed have zero probability. □

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References

[1] Armitage, D. H. and Gardiner, S. J. (2002) *Classical Potential Theory*, Springer.

[2] Benko, D., Dragnev, P. and Totik, V. (2012) Convexity of harmonic densities. *Rev. Mat. Iberoam.* **28** 1–14.
[3] Billingsley, P. (1968) *Convergence of Probability Measures*, Wiley.
[4] Courant, R., Friedrichs, K. and Lewy, H. (1928) Über die partiellen Differenzengleichungen der mathematischen Physik. *Math. Ann.* **100** 32–74.
[5] Doob, J. L. (1984) *Classical Potential Theory and its Probabilistic Counterpart*, Springer.
[6] Garnett, J. B. and Marshall, D. E. (2005) *Harmonic Measure*, New Mathematical Monographs, Cambridge University Press.
[7] Guy, R. K., Krattenthaler, C. and Sagan, B. E. (1992) Lattice paths, reflections, and dimension-changing bijections. *Ars Combin.* **34** 3–15.
[8] Kallenberg, O. (1997) *Foundations of Modern Probability*, Probability and Its Applications, Springer.
[9] Port, S. C. and Stone, C. J. (1978) *Brownian Motion and Classical Potential Theory*, Probability and Mathematical Statistics, Academic.
[10] Ransford, T. (1995) *Potential Theory in the Complex Plane*, Cambridge University Press.